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HYBRID RHEOLOGICAL MODEL FOR PREDICTING NON-NEWTONIAN TURBULENT PIPE FLOW

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The Wilson-Thomas model for turbulent flow of Bingham plastic fluids has been extended recently to predicting the transition velocity based on a Bingham rheogram. In practice most slurry rheograms do not follow the Bingham model, but are curved (convex upwards) at low strain rates. Alternate models such as the yield-power law, provide a good fit to the low-strain-rate data, they tend to underestimate apparent viscosities at high strain rates. The current paper considers a hybrid rheological model consisting of a cubic spline fit to the low-strain-rate data merging into a Bingham linear model above a limiting strain rate. The Wilson-Thomas turbulent flow analysis is extended to the hybrid model. It is found that the turbulent prediction depends only on the area difference between true Bingham rheogram and the cubic spline rheogram rather than the actual area distribution. The turbulent flow friction factor predictions for the hybrid model are compared with the true Bingham model predictions. Differences are discussed, and predictions using the hybrid model are compared with turbulent flow measurements from the literature.

KEYWORDS: non-Newtonian turbulent flow, rheology

1. PROBLEMS WITH RHEOLOGICAL MODEL SELECTION

In calculating pressure drops in pipelines carrying non-Newtonian fluids, it is often necessary to extrapolate to values of shear stress considerably in excess of those on the rheogram. Although rheograms are very often curved (convex upward) at low strain (or shear) rates, they often tend to approximate straight lines at high strain rates as is illustrated later in Figure 1.

The Wilson-Thomas prediction method for tubulent flow of Bingham plastic fluids, first published in 1985, has been extended recently to prediction of the transition velocity (2006) and from smooth to rough walled pipes (2007). All these previous analyses are based on an assumed true Bingham rheogram with a yield stress and a linear section of slope equal to the plastic viscosity. i.e. these previous analyses do not allow for the curved sections of the rheograms at low strain rates. Whilst alternate models such as the yield power law, provide a good fit to the low strain rate data they tend to physically unrealistic apparent viscosities less than water at high strain rates.

2. FUNDAMENTALS OF WILSON-THOMAS APPROACH

Before considering the proposed hybrid model it is worthwhile reviewing the fundamentals of the Wilson-Thomas method applied to Bingham plastics. For turbulent flow this method involves the sub-layer thickening factor α and a profile-blunting term Ω. The important factor is α, which is applied to the model of Newtonian turbulent flow. This model is considered through its "engineering" approximation to the velocity distribution, used for a smooth boundary. Here the effect of fluid viscosity is confined to a thin sub-layer which extends from the wall to a distance δ given by:

$$\delta = 11.6\ \mu / \rho\ U^* \tag{1}$$

where μ and ρ are the viscosity and density respectively, of the Newtonian fluid and U^* is the shear velocity at the wall. In the main flow, ($y > \delta$) it is assumed that all momentum transfer takes place by turbulent mixing, which is an inertial process rather than a viscous one. It follows that the velocity profile in the main flow is logarithmic, as opposed to the increase in the linear sub-layer where

$$u = \tau_0\ y / \mu \tag{2}$$

As the velocity gradient in the viscous sub-layer is much steeper than that in the logarithmic zone, the sub-layer has an influence on mean velocity which is disproportionate to its small thickness. If a non-Newtonian fluid is to be substituted for the Newtonian one, the momentum transport in the logarithmic zone is not affected by Newtonian viscosity, and the same should apply for non-Newtonian rheological properties. [In the part of the flow nearest the pipe axis, some change in the velocity profile results from non-zero τ_y; this is included in the velocity-blunting term Ω as shown by Wilson & Thomas (1985)].

Within the sub-layer, which occupies a small portion of the flow where variations in shear stress and velocity gradient are negligible, laminar conditions are approached. Here a linear velocity increase equivalent to Eq. 2 should also apply for a non-Newtonian fluid, but the measure of the thickness of the viscous sub-layer may exceed 11.6. This point is related to a conceptual model proposed by Lumley (1973, 1978) to explain the phenomenon of drag reduction in aqueous flows, which results from the addition of small quantities of certain long-chain molecules. The size of the larger turbulent eddies (the macro scale of turbulence) is directly proportional to the distance from the wall, while the size of the smallest eddies (the dissipative, or Kolmogorov scale) does not change much with this distance. At large values of y the inertial macro-eddies are much bigger than the dissipative micro-eddies, and between them is a whole cascade of turbulent eddy sizes, but as the wall is approached, the range of possible eddy sizes shrinks until the size of the largest and smallest eddies are equal, indicating the elimination of turbulence. At this point y equals δ, the thickness (in a statistical sense) of the viscous sub-layer. The Wilson-Thomas model for non-Newtonians was developed on this basis, with the interaction of the eddies in turbulent flow causing 'vortex stretching' that produces abrupt increases in strain rates and in the viscous dissipation of the smallest eddies. As the energy available for dissipation is fixed by the turbulent energy cascade, the result is an increase in the micro-eddy size, and a proportional increase in α. As shown by Wilson and Thomas (1985), the value of the mean velocity V_m which results from this thickened viscous sub-layer is given by Eq. 3 where V_N is the mean velocity for the equivalent smooth-wall flow of a Newtonian fluid with viscosity μ. Note that V_m/U^* can be written as $\sqrt{(8/f)}$ where f is the Darcy-Weisbach friction factor.

$$V_m/U^* = V_N/U^* + 11.6\,(\alpha - 1) - 2.5 \ln \alpha - \Omega \qquad (3)$$

3. HYBRID RHEOLOGICAL MODEL

The hybrid model involves a linear rheogram at high shear rates (above the limiting values γ'_L and τ_L) and a non-linear one at lower values as indicated in Figure 1. (Note: Figure 1 refers to a specific hybrid model based on the cubic spline, which along with the data points shown, is discussed later. The general form shown still applies to the current discussion). For cases where the near-wall values of γ' and τ exceed the limiting values, the rheogram can be taken as that of a Bingham plastic, based on (extrapolated) yield stress τ_B and slope η_B. The Wilson-Thomas (1985) analysis of turbulent flow of a Bingham plastic is based on α, the ratio defined in connection with Eq.3 and equal to the ratio of the area below the Bingham line to that below the equivalent Newtonian triangle, matching at τ_0.

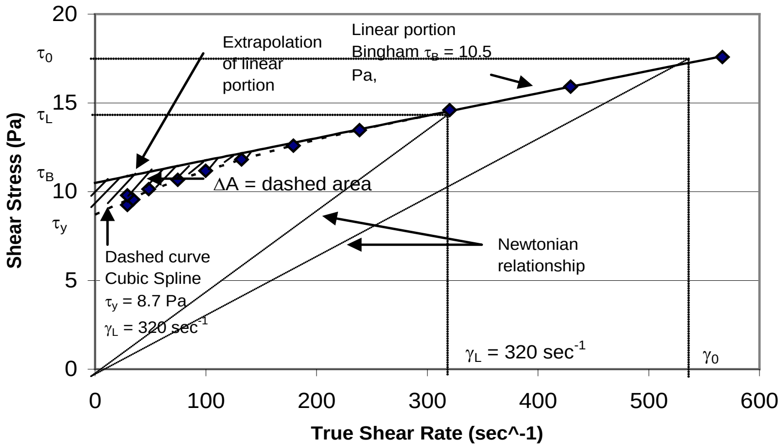

Figure 1 Cubic spline hybrid model applied to rotational viscometer data

The rheogram falls below the Bingham line at $\tau = \tau_L$. The area change ΔA below the Bingham line is defined as equal to $\Phi[(\tau_L - \tau_B)\gamma'_L/2]$. The next step is to obtain an expression for α in terms of the dimensionless ratio Φ. Following earlier work, ξ is defined as τ_B/τ_o, where τ_B is the intercept to the Bingham line and τ_o as the wall shear stress. For convenience λ will be defined as τ_L/τ_o. Thus $\alpha = 2A/(\tau_o\gamma'_o)$, where A is the area beneath the rheogram. The resulting expression for α is:

$$\alpha = 2(\tau_o\gamma'_o)/(\tau_o\gamma'_o) - (\tau_o - \tau_B)\gamma'_o/(\tau_o\gamma'_o) - 2\Delta A/(\tau_o\gamma'_o) =$$
$$= 2 - (1 - \xi) - \Phi[(\tau_L - \tau_B)\gamma'_L]/(\tau_o\gamma'_o)$$

or:

$$\alpha = 1 + \xi - \Phi(\lambda - \xi)(\gamma'_L/\gamma'_o) \qquad (4)$$

At this point, the use of similar triangles indicates that:

$$\gamma'_L/\gamma'_o = (\tau_L - \tau_B)/(\tau_o - \tau_B) = (\lambda - \xi)/(1 - \xi)$$

Substitution than gives the basic formula for α as follows:

$$\alpha = 1 + \xi - \Phi(\lambda - \xi)^2/(1 - \xi) \qquad (5)$$

It is important to note that, although α depends on Φ, and thus on ΔA, it does not matter how the area ΔA is distributed through the left portion of the rheogram.

From Wilson and Thomas (1985) for the hybrid rheological model the velocity blunting term Ω in eqn 3 becomes:

$$\Omega = -2.5 \ln[(\tau_B/\tau_y)/\xi - 1)(\tau_y/\tau_B)\ \xi\ ] - 2.5[(\tau_B/\tau_y)/\xi + 0.5]\ (\tau_y/\tau_B)^2\ \xi^2 \qquad (6)$$

4. THE BINGHAM-SPLINE RHEOLOGIC MODEL

The preceding section has dealt with a generalized curve leading from the Bingham line at low shear rates, and the conditions where the wall strain rate exceeds the upper limit of the curved section of the rheogram, so that the details of the curve are no longer needed, only the area change from the Bingham line. It is now appropriate to examine a specific type of curve, and for this it is proposed to use a specific function that has a smooth and seamless transition to the Bingham line at γ'_L. A suitable function that is often used in curve fitting is the cubic spline. Before dealing with the spline itself, it should be recalled that the Bingham segment ($\gamma' > \gamma'_L$ is represented by:

$$\tau = \tau_B + \eta_B\, \gamma' \qquad (7)$$

In the range $\gamma' < \gamma'_L$, there is more than one way to express a cubic spline, but for present purposes the most convenient is:

$$\tau = \tau_B + \eta_B\, \gamma' - (\tau_B - \tau_y)\{[1-(\gamma'/\gamma'_L)]^3\} \qquad (8)$$

This gives $\tau = \tau_B + \eta_B\, \gamma'_L$ at γ'_L and $\tau = \tau_y$ at $\gamma' = 0$. ΔA can be obtained by integration, yielding for ΔA and Φ:

$$\Delta A = 0.25(\tau_B - \tau_y)\, \gamma'_L \qquad (9)$$

$$\Phi = 0.5(1 - \tau_y/\tau_B)/(\tau_L/\tau_B - 1) \qquad (10)$$

As an example the cubic spline hybrid model has been applied to the rheogram shown in Figure 1. The data were obtained in a rotational viscometer with a bob to cup diameter ratio of 0.947 on a mineral sands tailings slurry. The true shear rate data shown were obtained using the method of Krieger and Elrod (1953) from numerical differentiation of the test data. The dashed curve in Figure 1 represents a cubic spline with τ_y = 8.7 Pa and γ'_L = 320 sec^{-1} and is seen to provide a reasonable fit to the data. For γ'_L > 320 sec^{-1} the data are described by the Bingham model with τ_B = 10.5 Pa and η_B = 0.0126 Pa.s.

5. TURBULENT FLOW PREDICTIONS

Figure 2 shows turbulent flow predictions for the cubic spline hybrid model for the cubic spline parameters shown in Figure 1 and Hedstrom numbers He = 1E5 and 1E7. Also shown for comparison are predictions based on the extrapolated Bingham model. Both laminar and turbulent flow predictions are shown for the Bingham model. The dashed curve represents the friction factor for a Newtonian fluid. For the slurry properties shown in Figure 1, He = 1E5 and 1E7 relate to pipe diameters of 37.5 mm and 375 mm respectively.

It is obvious from Figure 2 that there is very little difference between the turbulent flow predictions using the cubic spline model and the Bingham model. Consideration of Figure 1 shows this is to be expected. Any difference between the cubic spline hybrid model and the Bingham model depends on the area difference ΔA and this is small for the data of Figure 1. Also it is clear from Figure 1 that as τ_0 becomes larger the relative importance of ΔA decreases. Under turbulent flow conditions the value of τ_0 increases rapidly meaning that the relative importance of ΔA rapidly decreases.

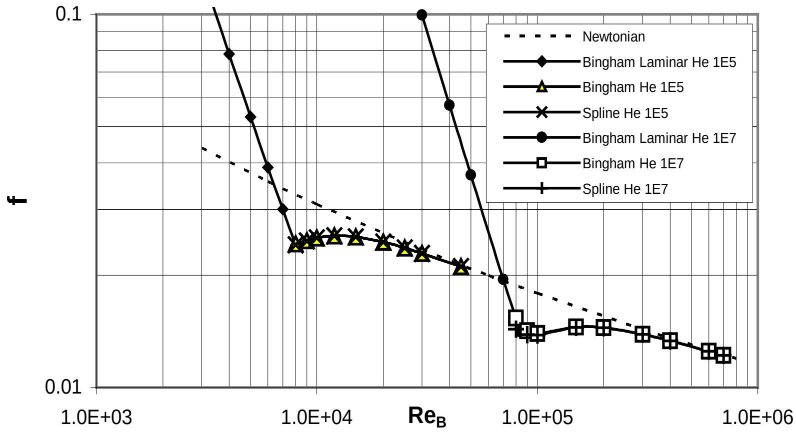

Figure 2 Cubic spline and Bingham predictions for He 1E5 and He 1E7

The Figure 1 rheogram data are typical of many results obtained in rotational viscometers. Comparison of cubic spline hybrid and Bingham predictions in Figure 2 indicates little difference in predictions. These results therefore suggest that ignoring the low shear rate curved data and basing predictions on the Bingham model fitted to the high shear rate data will usually provide adequate accuracy.

Let us now consider some pipe-flow kaolin data of Slatter and van Sittert (1999) which they analysed using the yield-power-law model. Tabulated laminar flow data in a 28.34 mm smooth walled pipe was provided by Slatter (2006) and the data has been differentiated to obtain shear stress versus true shear rate with the result shown in Figure 3. Yield-Power-Law (YPL), Bingham and hybrid cubic spline models are shown fitted to the data. Slatter and van Sittert used the same K and n values for their YPL fit but a higher yield stress of 5.505 Pa. The higher yield stress may have been an averaged fit to results from a range of pipe sizes.

Note how the equivalent plastic viscosity of the YPL model has reduced to 0.0034 Pa.s at the maximum 1400 sec^{-1} shear rate shown (shear stress 13 Pa). The equivalent plastic viscosity reduces to the viscosity of water at a shear stress of just 25 Pa and the effective (secant) YPL viscosity reduces to that of water at a shear stress of 47 Pa which equates to a velocity of about 4.1 m/s in the 28.34 mm pipe. This illustrates the extrapolation problem with the YPL model which the hybrid cubic spline model overcomes

Figure 4 shows predicted friction factor versus Reynolds number using the Bingham and hybrid models for comparison with the measured data. The Bingham and hybrid models both predict the turbulent flow data well. Also shown in Figure 4 are predictions based on the YPL model. The Bingham Reynolds number Re_B does not strictly apply to the YPL model but for comparison in Figure 4 the predicted YPL friction factor for velocities relevant to the range of turbulent flow data are plotted against Re_B. Prediction A is based on the Thomas-Wilson (1987) method. Prediction B assumes a Newtonian model with viscosity equal to the effective YPL viscosity at the relevant wall shear stress. It is clear that the YPL predictions predict the wrong trend relative to the data. This is because of the continuous reduction in YPL effective viscosity as the wall shear stress increases discussed previously in relation to Figure 3. At the highest data point (velocity 3.695 m/s, Re_B = 2.37E4) the effective YPL viscosity has reduced to less than half of the Bingham plastic viscosity. The Thomas-Wilson (1987) YPL analysis assumed the YPL model applied up to the maximum shear stress of interest in turbulent flow. It is clear that this is not the case for this kaolin data.

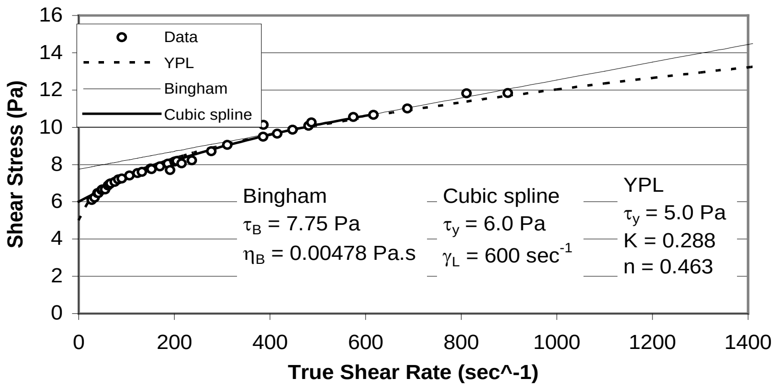

Figure 3 Kaolin data of Slatter and van Sittert

6. CONCLUSION

The Wilson-Thomas non-Newtonian turbulent flow prediction method has been extended to a hybrid cubic-spline-Bingham rheological model. Predictions based on typical rotational viscometer rheological data indicate little difference between predictions using the hybrid model and the Bingham model. It is therefore concluded that the low shear rate curved data can be ignored and a Bingham model fitted to the high shear rate data provides a remarkably good turbulent flow prediction.

Predictions based on literature data of Slatter and van Sittert (1997) illustrate the advantages of the cubic spline model compared with the Yield Power Law model.

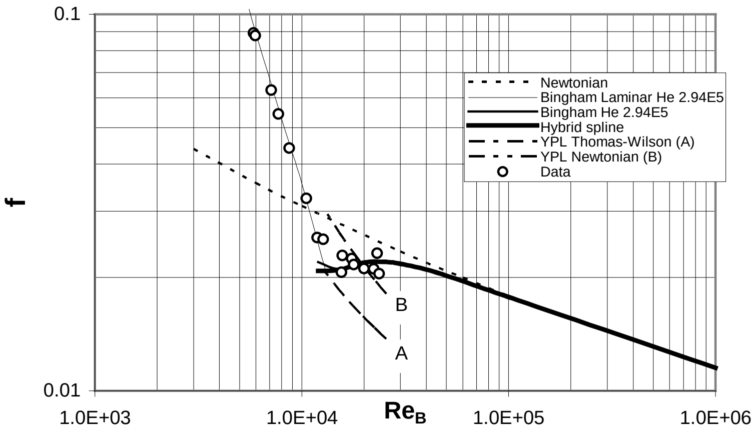

Figure 4 Turbulent flow predictions for kaolin in 28.34 mm pipe

Once again the Bingham model fitted to high shear rate data provides an adequate turbulent flow prediction. The wall shear stress applying to turbulent flow conditions will generally be far higher than the maximum shear stress obtainable for the laminar flow rheogram. YPL extrapolation will therefore always be uncertain and most likely will underpredict the effective viscosity for turbulent flow. It is therefore recommended that a Bingham model (or hybrid cubic spline) be fitted to the highest shear rate data available, rather than use a YPL model. The present paper suggests that ignoring the low shear rate curved portion of the YPL rheogram and using the Bingham model results in tolerable error. The error could be reduced further by using the hybrid cubic spline model.

NOMENCLATURE

A Area under rheogram
f Friction factor (Darcy-Weisbach)
He Hedstrom number = $D^2\tau_B\rho/\eta_B^2$
K Yield Power Law constant
n Power law exponent
u Velocity
Re_B Bingham plastic Reynolds number
U* Shear velocity = $\sqrt{(\tau_w/\rho)}$
V_N Mean velocity for equivalent Newtonian fluid
V_m Mean velocity
y Distance from pipe wall
α Ratio area under non-Newtonian rheogram to area under Newtonian rheogram
ΔA Area under Bingham rheogram – area under actual curved rheogram
δ Thickness of viscous sub-layer
γ' Strain rate
γ'_0 Strain rate at wall
γ'_L Limiting strain rate cubic spline model
η_B Bingham plastic viscosity
λ τ_L/τ_B
μ Secant viscosity
ξ τ_B/τ_0
ρ Slurry density
τ_0 Wall shear stress
τ_B Bingham yield stress
τ_y Yield stress
τ_L Shear stress at γ'_L
Φ Area change below Bingham
Ω Profile blunting term

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